A quick run through of 180 years of Classical Mechanics – for better appreciation of Quantum Mechanics

- For better appreciation of quantum mechanics
- You might have learned all the stuffs, but often *looked too deep into the technical details* and *care too much on exams*, so much so you could have lost the big picture/and interest!
- *For a cultural linking up of what you learned in different courses*
- An excellent example of *how physics develops*

For those who followed the course pattern of University Physics I and Classical Mechanics I or equivalents, this will largely be a review. For those haven't taken Lagrangian and Hamiltonian Mechanics, don't worry – just open up your mind, relax and absorb.

- *Newton (1642 - 1727): Newtonian Mechanics*
- *Lagrange (1736 - 1813): Lagrangian Mechanics*
- *Hamilton (1805 - 1865): Hamiltonian Mechanics*

Essential concepts illustrated by one simple example – the harmonic oscillator

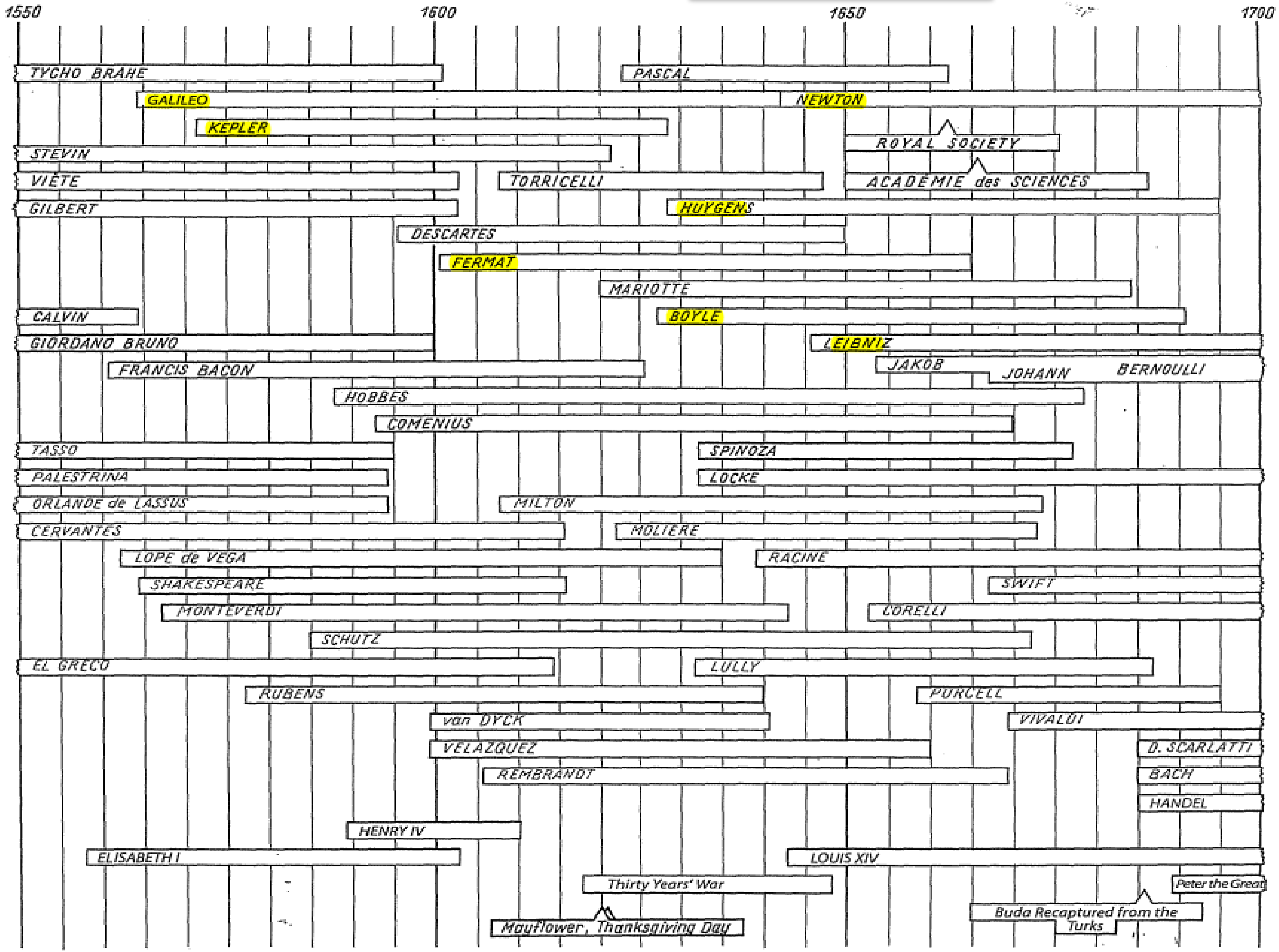
This part of the Chapter is a modified version of a talk by P.M. Hui to secondary school physics teachers in a Teacher's Program on 13 July 2017 organized by the Curriculum Development Institute, Education Bureau, HKSAR

I believe and you will see…

温故必然知新
“review always leads to new knowledge”

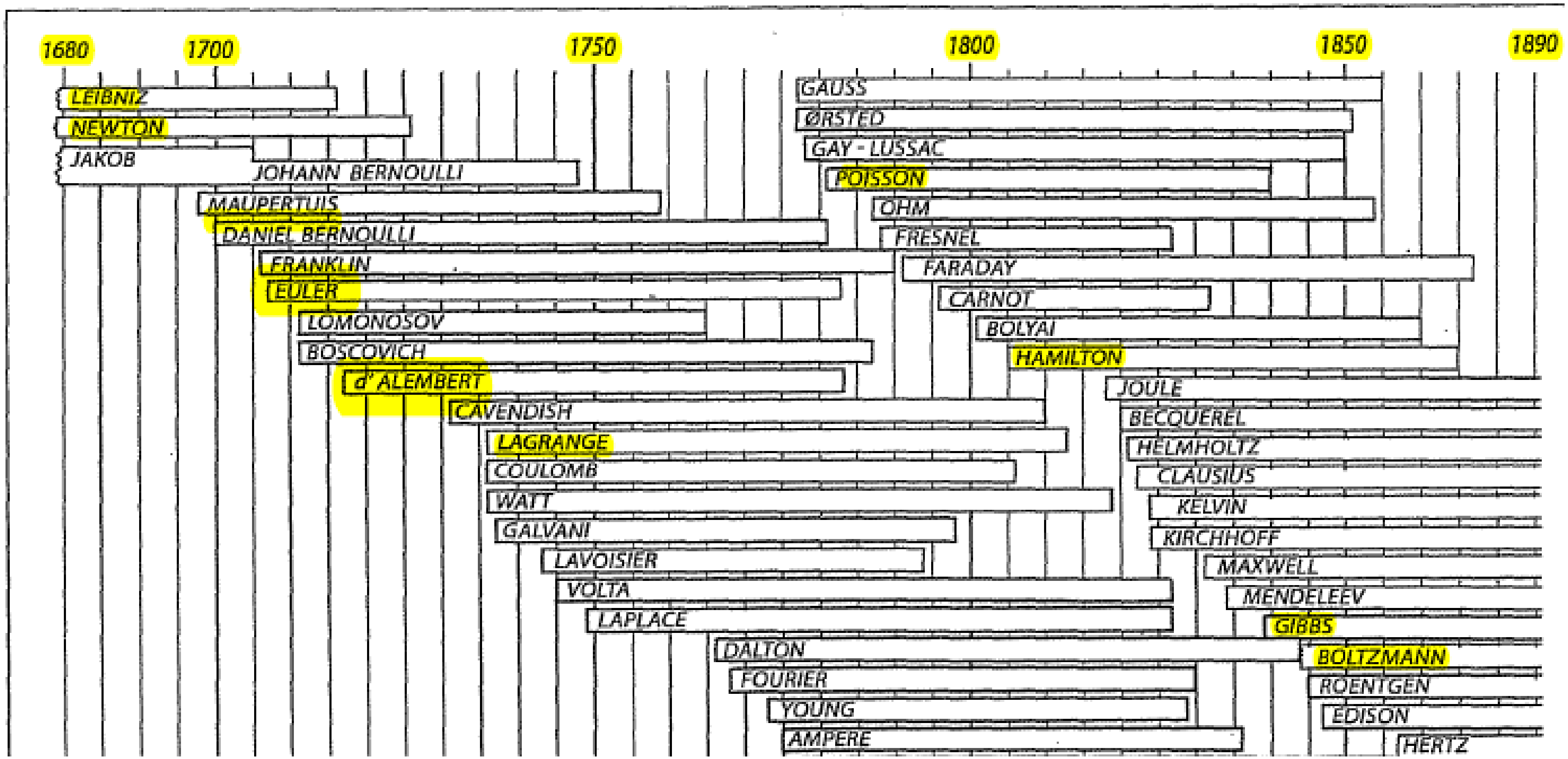
學習何必計較致用
“why care too much of what you learn is useful or not” when learning
[but one day, you will suddenly see the connections/applications]

Where were we?



Taken from: K. Simonyi, *A Cultural History of Physics* (CRC Press 2010). A wonderful book for those interested in what physics develops

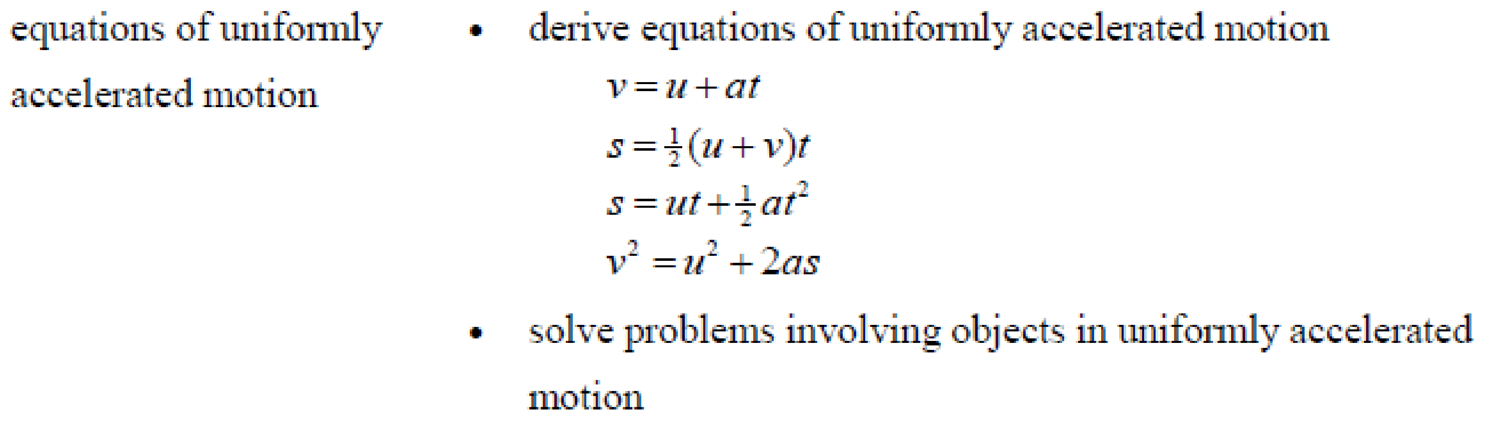
The Builders of Classical Physics



Taken from: K. Simonyi, *A Cultural History of Physics* (CRC Press 2010)

First Course on Classical Mechanics: Newtonian Mechanics

Kinematics: position, velocity, acceleration as time evolves



equations of uniformly accelerated motion	• derive equations of uniformly accelerated motion $v = u + at$ $s = \frac{1}{2}(u + v)t$ $s = ut + \frac{1}{2}at^2$ $v^2 = u^2 + 2as$ • solve problems involving objects in uniformly accelerated motion

Basically, given position *x* (now) and velocity *v* (now), and if for some reason an object has an acceleration, what are its next position and next velocity?

But doesn’t know (doesn’t care) why there is an acceleration

Newton's (second) Law

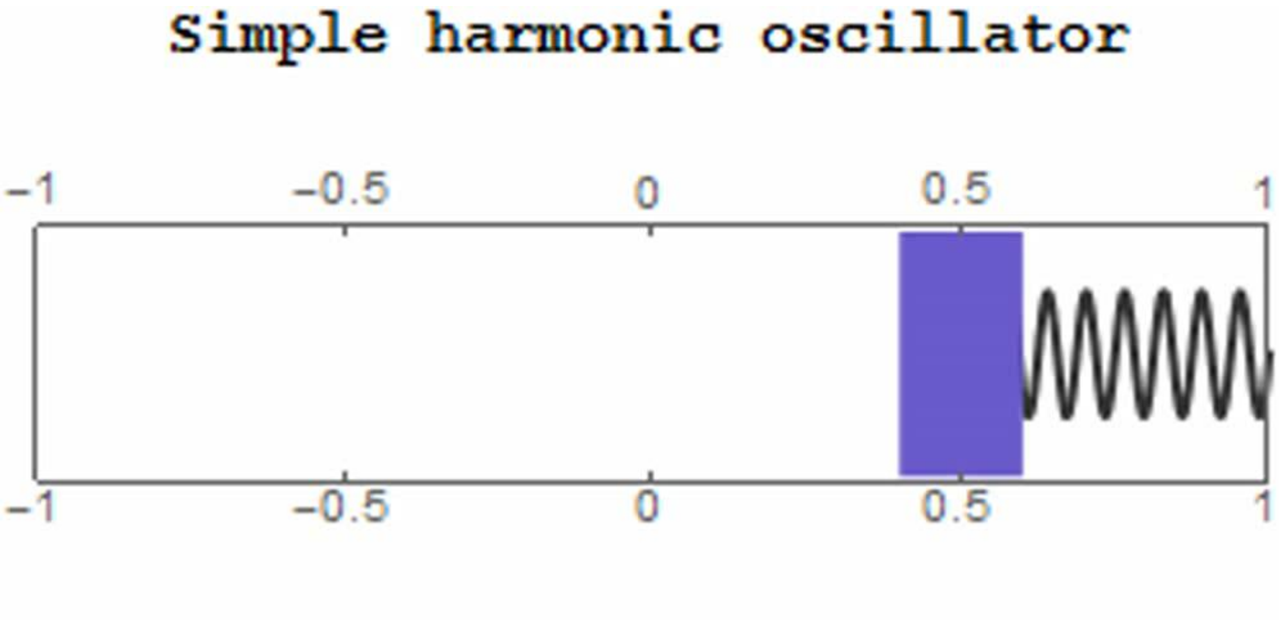
Isaac Newton (1642 – 1727)

$$F = m\underbrace{\overbrace{a}^{\text{acceleration}}}_{\propto\ \text{change in velocity}}$$

force leads to acceleration

$$\left[\text{contrast with } \underbrace{\text{kinematics}}_{\text{don't care how "a" comes about}}\right]$$

Pdf file doesn’t show the animations, see separate powerpoint file



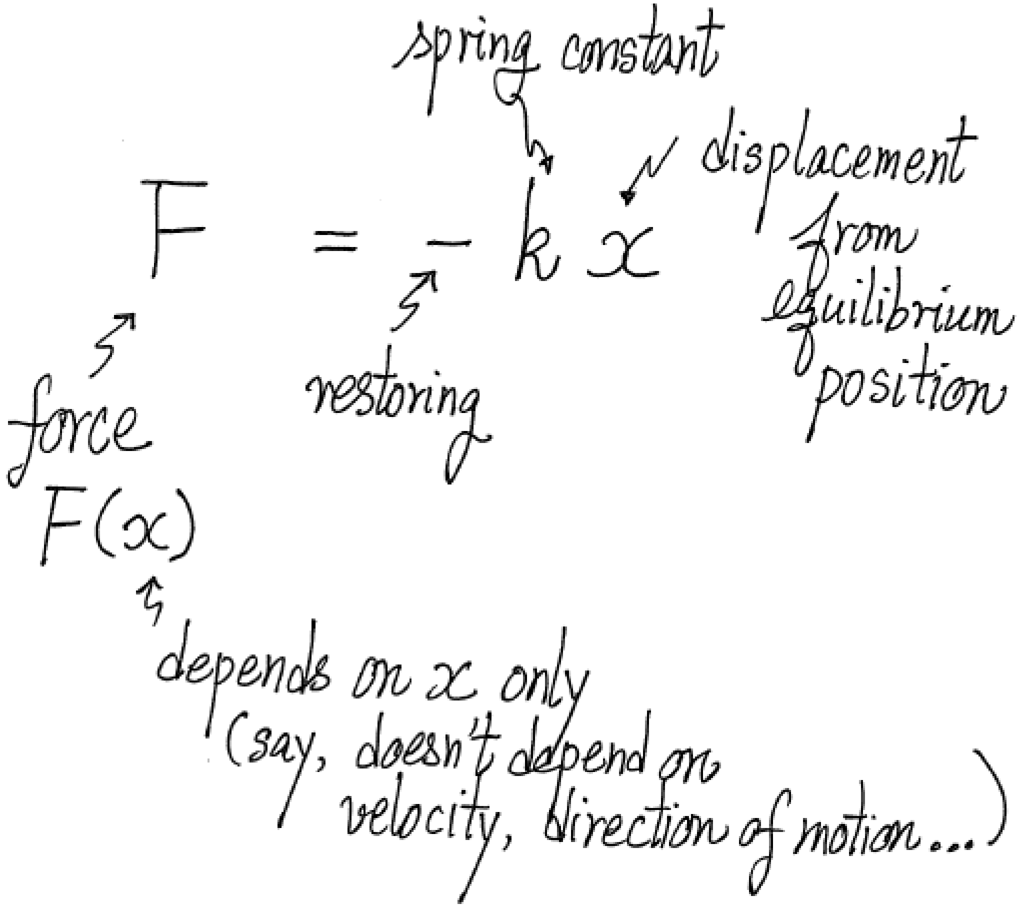
Amplitude:
A = 0.5
Spring constant:
k = 1
Mass:
m = 2

mass *m*
spring constant *k*
smooth floor
No air drag…

- Template: "Simple Harmonic Motion of a Spring“ from the **Wolfram Demonstrations Project** http://demonstrations.wolfram.com/SimpleHarmonicMotionOfASpring/
- Contributed by: Kenny F. Stephens II
- Same acknowledgment to other animations, unless stated otherwise

To apply Newton’s Law (e.g. oscillator) , we need to know the Force *F*

Hooke’s law (Robert Hooke 1635 - 1703) – assistant to Boyle



$$F = -kx$$

spring constant

displacement from equilibrium position

restoring

force

$F(x)$

depends on x only (say, doesn't depend on velocity, direction of motion...)

[Hooke, about 1660]
Hooke discovered the cells using his microscope and made many discoveries in different fields. For unusual reasons, he was eliminated from the history of science for almost 300 years!
See *The Forgotten Genius: The Biography Of Robert Hooke 1635-1703* by Stephen Inwood for Hooke’s story.

$$F = ma \quad \text{(Newton)}$$

$-kx$ (Hooke)

$$ma = -kx$$

$$\boxed{m\frac{d^2x}{dt^2} = -kx}$$

Equation of motion

The most important equation in solving mechanics problems and then you learned techniques of solving it

or

$$\boxed{\frac{d^2x}{dt^2} = -\frac{k}{m}x = -\omega^2 x}$$

acceleration $\sim \frac{\text{length}}{(\text{time})^2}$

$\omega^2 = \frac{k}{m}$ has units $\frac{1}{(\text{time})^2}$

length

ω is the angular frequency

$$\boxed{\frac{d^2x}{dt^2} = -\omega^2 x}$$ Equation of Motion (*)

What does the equation do?

- To solve for $x(t)$
- By inspection,

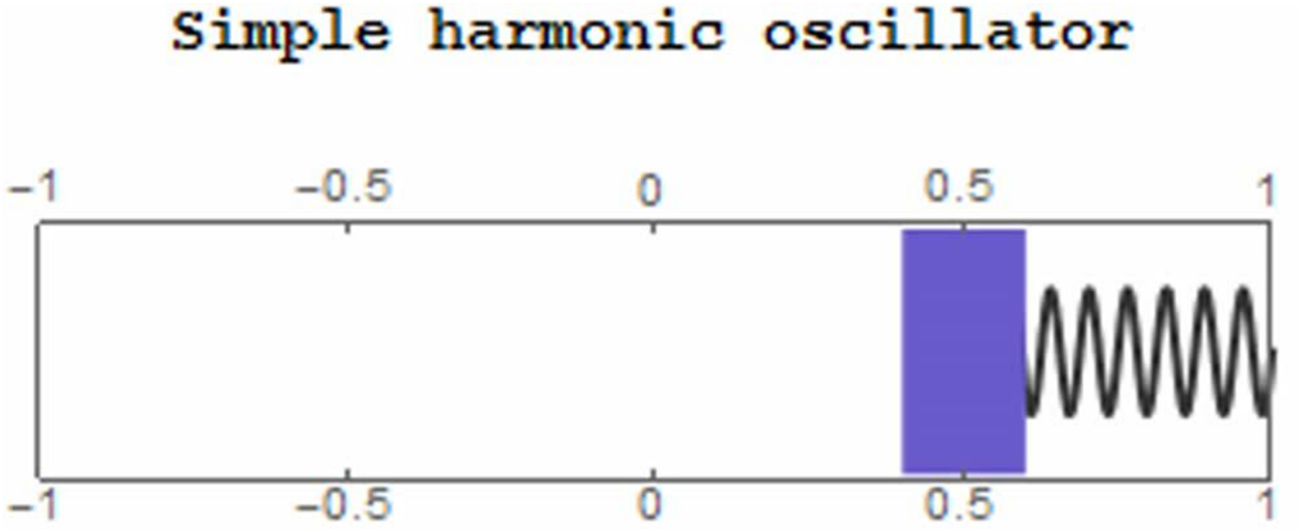
$$\boxed{x(t) = A\cos\omega t + B\sin\omega t}$$ works!

Concepts that follow... (don't worry)

- $x(t)$ is a real quantity
- linear superposition of solutions (linear homogeneous eq.)
- A, B to be determined by initial conditions

- E.g. $\underbrace{\text{Take mass to one side}}_{t=0,\ x(0)=\underbrace{A}_{\text{amplitude}}}$ and $\underbrace{\text{release it}}_{v(0)=0}$

$x(t) = A \cos \omega t$ is the solution.



Amplitude:
A = 0.5

Spring constant:
k = 1

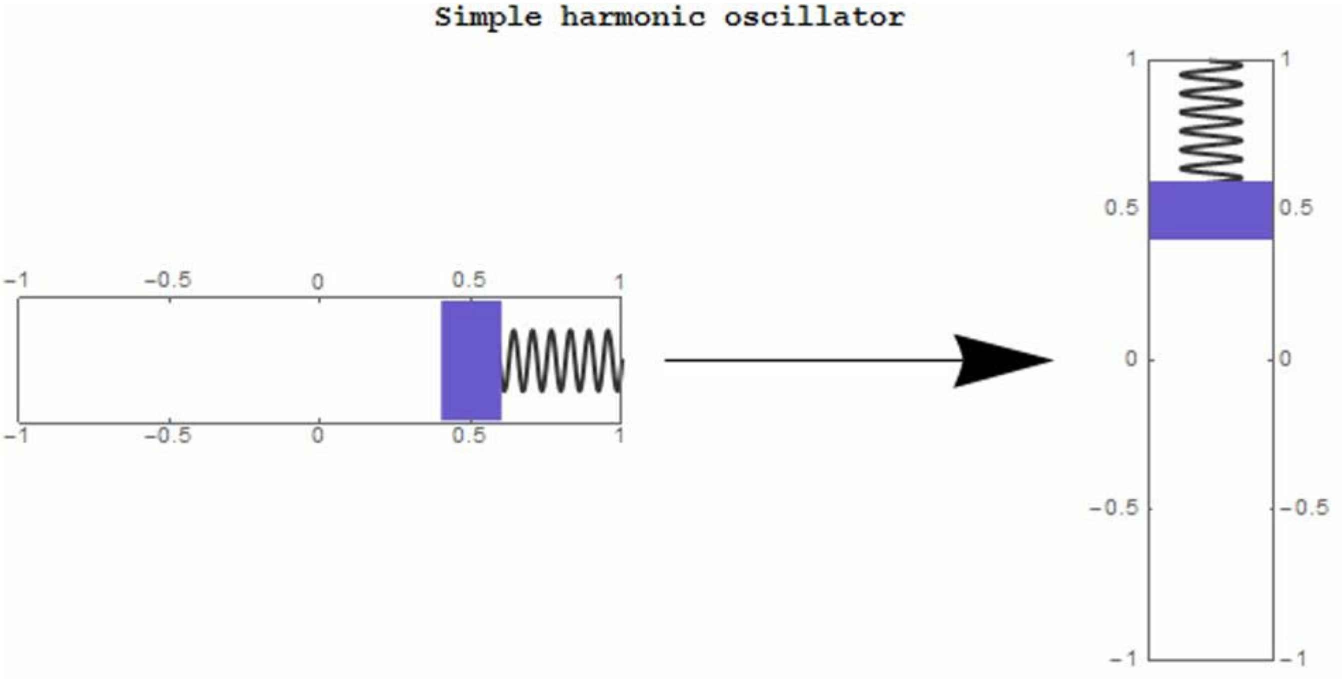
Mass:
m = 2

Motion:

$$x(t) = A\cos\left(\sqrt{\frac{k}{m}}t\right)$$

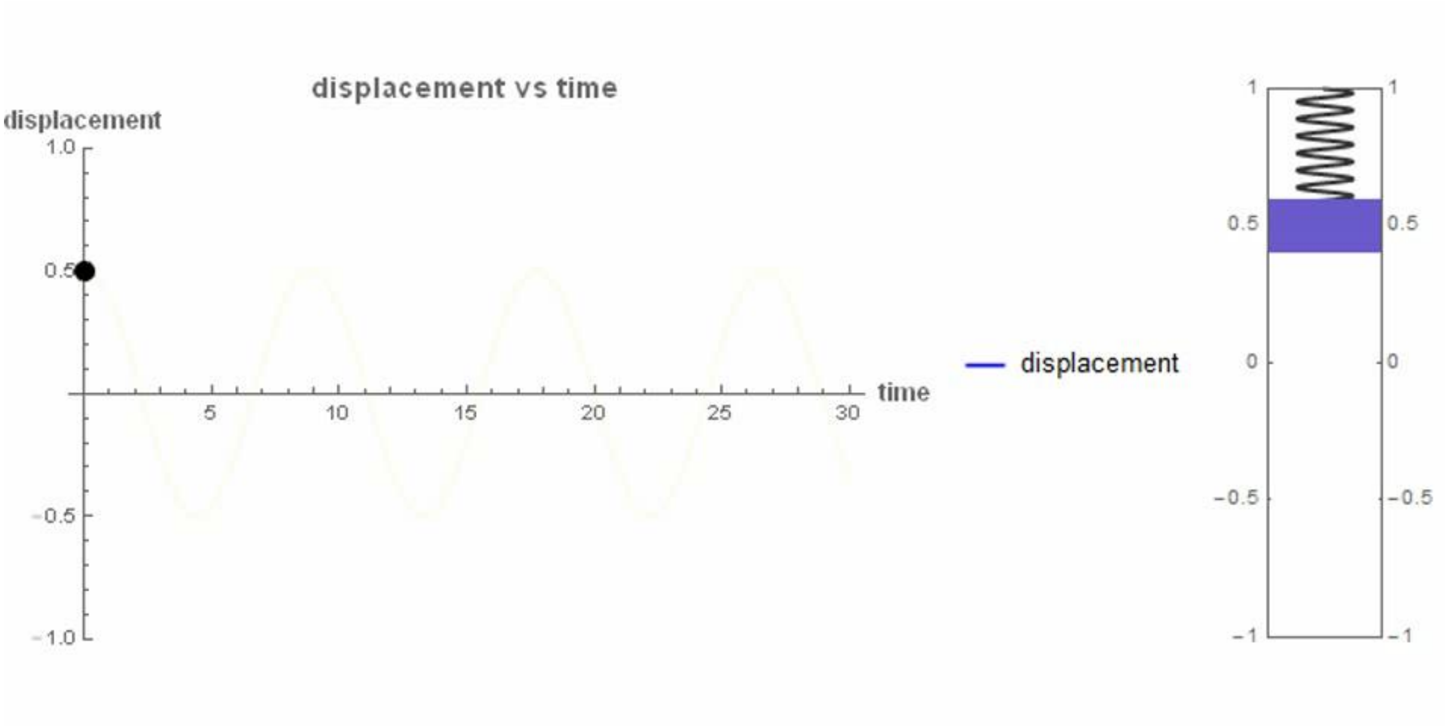
This is important! At this point, we could build a clock to measure time. Indeed, Hooke invented a watch. Huygens, Hooke's contemporary, also invented one. And they fought over priority!

For convenience, sometimes the oscillation is displayed vertically instead without worrying about "g"



Displacement vs time

For convenience (to save some space), the oscillation is displayed vertically instead without worrying about “g”



Amplitude:
A = 0.5

Spring constant:
k = 1

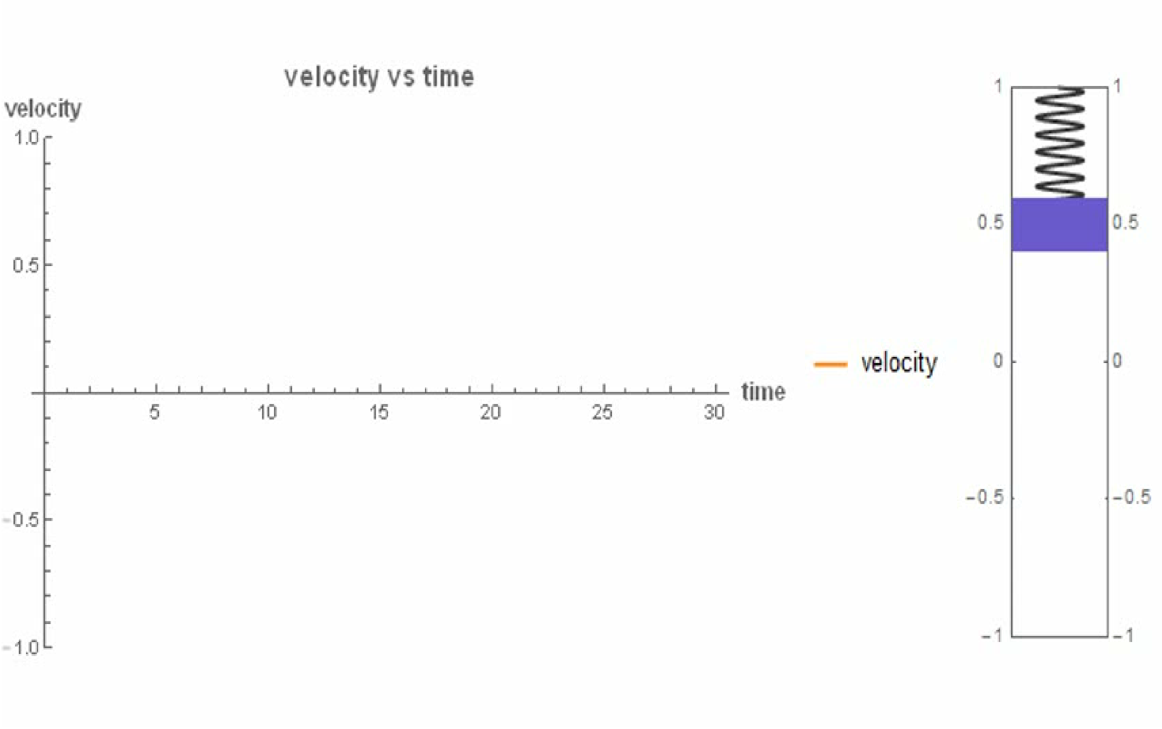
Mass:
m = 2

Motion:

$$x(t) = A\cos\left(\sqrt{\frac{k}{m}}t\right)$$

How about v(t)?

Harmonic oscillator (velocity vs time)



[Animation credit: LEUNG Chun Hei (MPhil Student, CUHK)]

The big picture:

Newton’s Law gives the

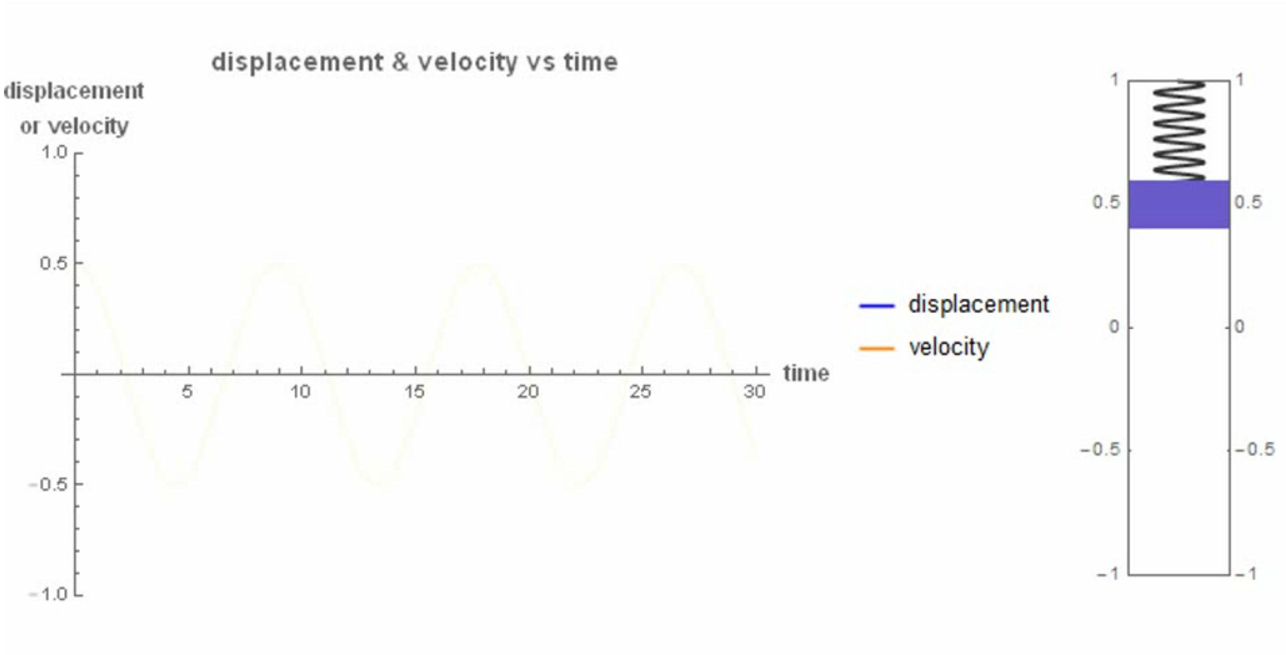
Equation of Motion

from which sequential motion can be solved

Finding the right **Equation of Motion** or a **systematic way** of getting the **Equation of Motion** has been the central business of physics for centuries!

Work-Energy Theorem

Harmonic oscillator (displacement & velocity vs time)



Inspect:

- Ranges of time when **mass movement and F(x) are in the same direction** => **speeding up** of the mass
- Ranges of time when **mass movement and F(x) are in opposite directions** => **slowing down** the mass
- Question: Any general statement?

Work-Energy Theorem

What is the question?

- Now, force acting on a body changes its velocity. In our simple 1D case, the force speeds up or slows down the mass.
- *Relation between the change in speed or a related quantity when* ***force*** *acts on the mass for some time and the* ***distance*** *that the mass has travelled in that time?*
- *Here comes* ***kinetic energy*** *and work done!*

Simplest case : Constant acceleration a [not the oscillator case]

Kinematics
$$\boxed{v^2 - \underset{\text{initial}}{v_0^2} = 2a(\underbrace{x - \underset{\text{initial}}{x_0}}_{\text{distance travelled } d})}$$

Newton : $a = \frac{F}{m}$

$$\therefore \quad v^2 - v_0^2 = 2\frac{\overset{\text{assumed constant}}{F}}{m}d$$

$$\Rightarrow \underbrace{\left(\frac{1}{2}mv^2\right)}_{} - \underbrace{\left(\frac{1}{2}mv_0^2\right)}_{} = F\cdot d$$

$$\underset{\text{kinetic energy}}{K_2} - \underset{\text{initial kinetic energy}}{K_1} = \underbrace{F\cdot d}_{\text{Work done by the force}}$$

Speeding up the mass as reflected in increase in kinetic energy

Force and motion in same direction, Product (work done) is positive

What if F is $F(x)$ [not a constant]?

- Easy, make the argument bit by bit

$$\underbrace{\left(\frac{1}{2}mv_2^2\right)}_{K_2} - \underbrace{\left(\frac{1}{2}mv_1^2\right)}_{K_1} = \underbrace{\int_{x_1}^{x_2} F(x)\,dx}_{W \leftarrow \text{Work done by force}}$$

- More generally,

$$\int \vec{F}(\vec{x}) \cdot d\vec{x} \quad \text{(2D or 3D)}$$

- Don't Worry! The point is that energy enters and the physics is transparent!

$$\underbrace{\left(\frac{1}{2}mv_2^2\right)}_{K_2} - \underbrace{\left(\frac{1}{2}mv_1^2\right)}_{K_2} = \underbrace{\int_{x_1}^{x_2} F(x)\,dx}_{W \leftarrow \text{Work done by force}}$$

The big picture - The Physics is simple and easy…

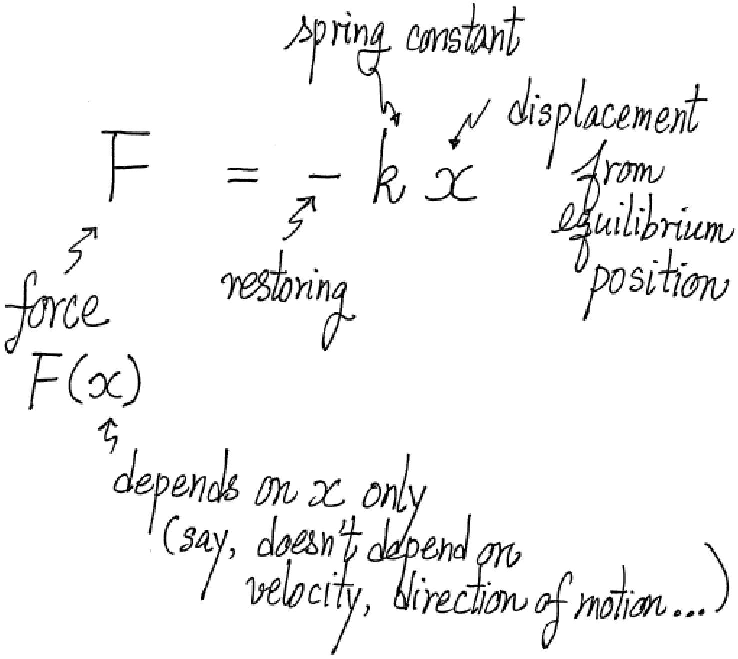
- *The force that moves the object does (something called) work*
- *The work serves to change the (something called) the kinetic energy of the object*
- *Make good sense!*
- *Energy enters into the picture*

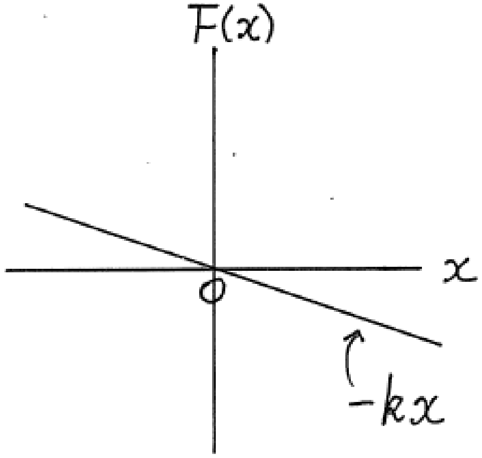
So far, good for any force.

Next, ***potential energy*** enters for a special kind of forces.

Potential Energy

- In physics, there is a type of force *F(x)* that depends only on the position *x* of the object, but *not* on the velocity or other stuffs
- Examples: Hooke's law and many restoring forces, Coulomb's law between charges, Newton's (gravitational) force between masses





Same force at *x*, regardless object's motion (to left or to right)

So what? Enters Potential Energy

If $F(x)$ can be expressed as $F(x) = -\frac{dV(x)}{dx}$,

(called **conservative forces**, just a name)

then the work done term becomes very simple!

$$W = \int_{x_1}^{x_2} \left(-\frac{dV}{dx} \right) dx = -\underset{\substack{\uparrow \\ \text{final}}}{V(x_2)} + \underset{\substack{\nwarrow \\ \text{initial}}}{V(x_1)}$$

Difference in potential energies

Example: $F(x) = -kx$, $V(x) = \underbrace{\frac{1}{2}kx^2}_{\text{"potential energy"}}$ works!

Counter-example: Friction doesn't work this way as it depends on direction of motion, among other stuffs!

Conservation of Energy

Work-Energy Theorem becomes… "potential energy"

$$K_2 - K_1 = W = -V(x_2) + V(x_1) = -V_2 + V_1$$

$$\Rightarrow \underbrace{K_2 + V_2}_{E_2} = \underbrace{K_1 + V_1}_{E_1}$$

Side Comment on QM:
In QM, starting point is
H = K + V

Conservation of (mechanical) energy

It requires a type of forces for which a potential energy function (of position) can be defined!

Ex: What is V for force between masses (charges)?
Ex: How to handle Lorentz force (force on charge due to magnetic field)?

The big picture –

- For a type of force $F(x)$ (typically depending only on position), ***work done*** can be expressed in terms of ***difference of potential energies***
- Mathematically, $F(x) = - dV/dx$ and thus defines $V(x)$
- Work-energy theorem then gives the law of ***conservation of (mechanical) energy***
- Often provides a convenient way to solve problems, other than solving the equation of motion (following motion in time)

Back to Harmonic Oscillator as example

(PE & KE & E vs time)

Kinetic energy $K = \frac{1}{2}mv^2$

Potential energy $V = \frac{1}{2}kx^2$

Total Energy $E = K + V$

- $\mathrm{M}ax. = \frac{1}{2}kA^2 = 0.125$

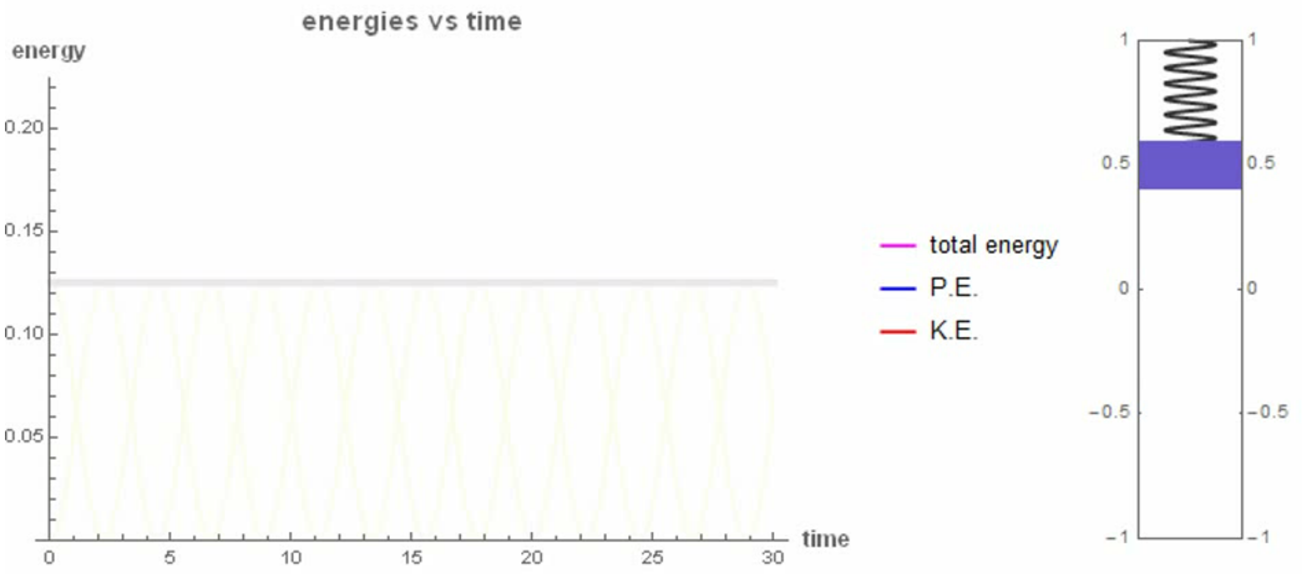
Amplitude:
A = 0.5

Spring constant:
k = 1

Mass:
m = 2

Motion:

$$x(t) = A\cos\left(\sqrt{\frac{k}{m}}t\right)$$



[Animation credit: LEUNG Chun Hei (MPhil Student, CUHK)]

Bonus – Relationship between Time averaged <K> and Time averaged <V> - *The virial theorem*

Harmonic Oscillator

$$\underset{\overset{\uparrow}{\frac{1}{2}\cdot\left(\frac{1}{2}kA^2\right)}}{\langle K\rangle} = \underset{\overset{\uparrow}{\frac{1}{2}\cdot\left(\frac{1}{2}kA^2\right)}}{\langle V\rangle} \qquad E=\frac{1}{2}kA^2$$

Generally, $V=\frac{1}{2}kx^{\text{\textcircled{2}}}=\frac{1}{2}kx^{n} \quad (n=2)$

$$\underbrace{\langle K\rangle = \frac{n}{2}\langle V\rangle}_{\text{Virial Theorem [Lagrange ~1770, Clausius~1870]}} = \frac{2}{2}\cdot\langle V\rangle = \langle V\rangle$$

Ex: What if $V(r)\sim\frac{1}{r}\sim r^{\text{\textcircled{-1}}}\sim r^{n}$?

[Have used the result in orbits and in Bohr's model!]

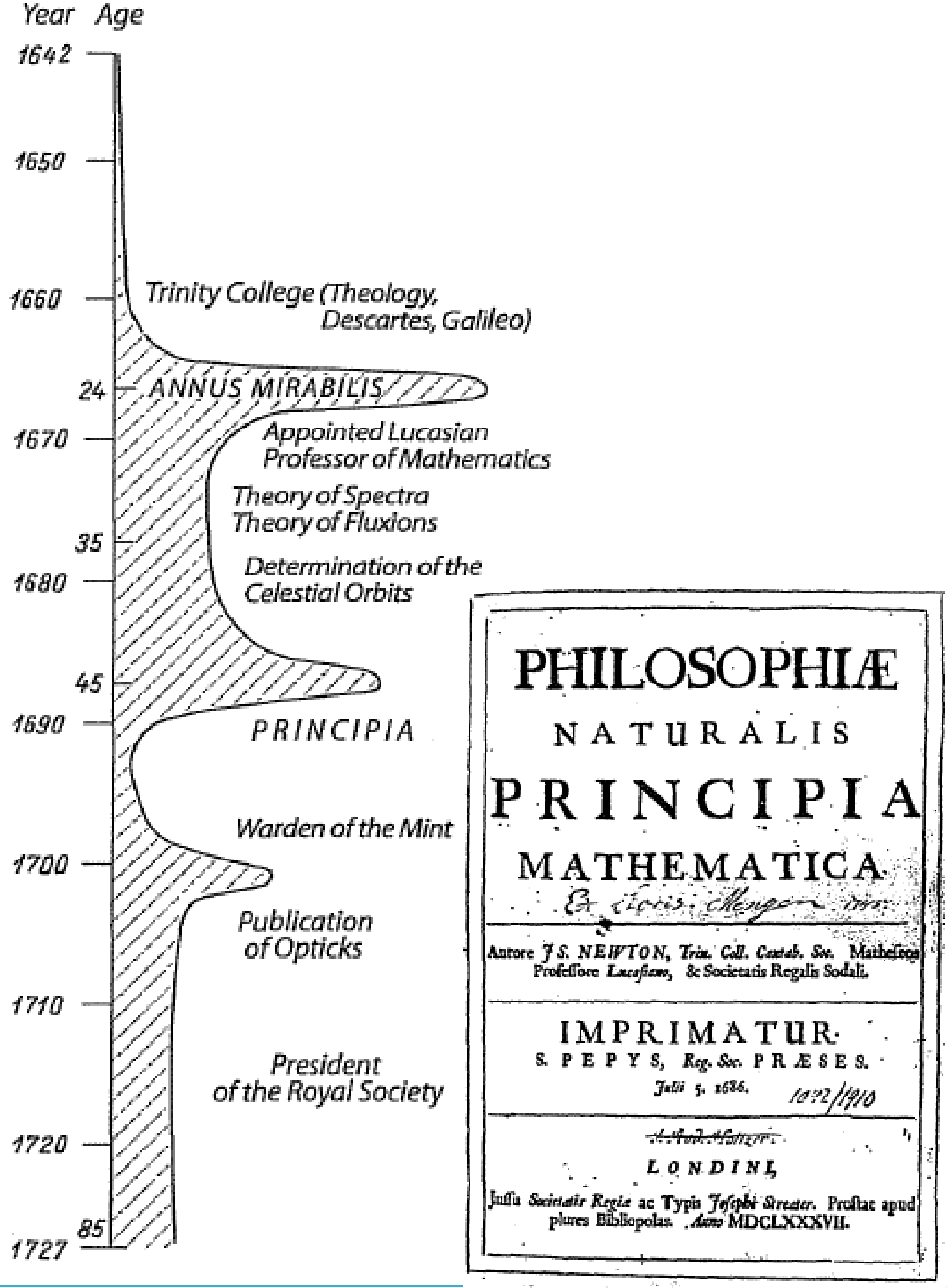
And that's physics – One answer to Many problems!

Let's meet Isaac Newton

Newton (1642 – 1727)

[National Portraits Gallery, London]

For Newton's story, see E. Dolnick, "The Clockwork Universe: Isaac Newton, the Royal Society, and the Birth of the Modern World" (2011).



PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA

Autore J S. NEWTON, Trin. Coll. Cantab. Soc. Matheseos Professore Lucasiano, & Societatis Regalis Sodali.

IMPRIMATUR.
S. PEPYS, Reg. Soc. PRÆSES.
Julii 5. 1686.
1072/1910

LONDINI,
Jussu Societatis Regiæ ac Typis Josephi Streater. Prostat apud plures Bibliopolas. Anno MDCLXXXVII.

[Taken from: K. Simonyi, *A Cultural History of Physics* (CRC Press 2010)]

Isaac Newton (1642 – 1727)

Newton lived a long 85 years. Born in 1642 after his father had died for 3 months, his unhappy childhood might have contributed to his difficult personality. When he was 19 years old in 1661, he entered Trinity College Cambridge, studying mathematics. There were the great works by Descartes and Galileo at that time. A few years later in 1665 (23 years old), England was hit by a severe (plague) epidemic, so severe that the whole University was closed for 2 years. Newton returned home and had a great time working in private. For Newton, 1665 and 1666 (bad time for England) turned out to be his best time – later people called these Newton's "miracle years". He was only 24 years old. In these two years, he worked out the ideas of the binomial theorem, differential calculus, theory of color, centripetal force, laws of motion, and theory of gravitation. What a time! The next year, he went back to Cambridge, worked on optics and made a reflecting telescope. In 1669, he was appointed the second Lucasian Professor of Mathematics at Cambridge, after Isaac Barrow, at the age of 27. He held the Chair Professorship for 33 years till 1702. The Lucasian Professor of Mathematics at Cambridge is now in its 19th generation. 200 years after Newton, it was occupied by Stokes, who gave us the Stoke's theorem and the Navier-Stoke equation in fluid mechanics. Another 80 years later, the Chair was taken by a young man who occupied it for 37 years. His name is Paul Dirac, who invented quantum mechanics and his Dirac equation for an electron. 300 years after Newton, there was Stephan Hawking. The current 19th Chair, Michael Cates, started his tenure in 2015 and he works on statistical mechanics of soft matter.

Newton liked to work in private and he hated his work being criticized by others. If he didn't publish, then people would have nothing to criticize! So he kept his work to himself. In 1672 (30 years old), he presented the "theory of light and colors" to the Royal Society (founded by Boyle and others) and his work was criticized. He hated that, and he hesitated and delayed the publication. Only 32 years later (61 years old), his book Opticks was published. For his greatest works of the 3 volumes of "Philosophia Naturalis Principia Mathematica" or "Principia" in short, he would not have published it if his friend Halley (the Halley of Halley's comet) had not pushed him to do so, worked out the publication arrangement with the Royal Society, and paid for the publication costs.

When he was 50 years old, he had a serious nervous breakdown. He recovered from it, but he could not make any significant contributions to science for his remaining 35 years. But he took on other roles. He become the warden of the Royal Mint, the office responsible for making coins in UK, in 1699; President of the Royal Society in 1703; knighted and became Sir Isaac in 1705.

Newton invented calculus. The German mathematician Leibniz said that he also invented calculus. Naturally, Newton didn't like it. In his position of President of the Royal Society, he made sure that his credit of inventing calculus was fully recognized. Ironically, it might be sad for Newton to see that the calculus and its notations in use today are those of Leibniz. Remember Hooke, who put forward the Hooke's law and the same Hooke who discovered the cell with his microscopes. Hooke made many discoveries. He even claimed priority in finding the theory of gravitation and that made Newton really angry. Hooke might be stupid to pick on Newton, the most powerful person at the time, as his enemy. Again, Newton made sure that Hooke would disappear from the history of science for 300 years! A story goes that Newton even made the only portrait of Hooke in the Royal Society disappear so that no one would know how he looked like! In recent years, historians give Hooke back some credits.

Over the years, many people wrote about Newton. A particular passage written by the English writer **Aldous Huxley** perhaps reflects the kind of person Newton was. Huxley wrote "*If we evolved a race of Isaac Newtons, that would not be progress. For the price Newton had to pay for being a supreme intellect was that he was incapable of friendship, love, fatherhood, and many other desirable things. As a man he was a failure; as a monster he was superb.*" This says rather deep about Newton. [PM Hui (July 2017)]

References:

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K. Simonyi, *A Cultural History of Physics*

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